

Modal Logic

$\diamond A$	=	It's possible that A.
	=	A is true in some possible world.
A	=	It's true that A.
	=	A is true in the actual world.
$\square A$	=	It's necessary that A.
	=	A is true in all possible worlds.

The result of writing “ \diamond ” or “ \square ,” and then a wff, is a wff.

A is possible (consistent, could be true) = $\diamond A$

A is necessary (must be true, has to be true) = $\square A$

A is impossible
(self-contradictory) = $\sim \diamond A$ = A couldn't be true.
= $\square \sim A$ = A has to be false.

A is consistent
(compatible) with B = It's possible that A and
B are both true.
= $\diamond(A \cdot B)$

A entails B = It's necessary that if A then B.
= $\square(A \supset B)$

A is a contingent
statement = A is possible and not-A
is possible.
= $(\diamond A \cdot \diamond \sim A)$

A is a contingent
truth = A is true but could have
been false.
= $(A \cdot \diamond \sim A)$

It's usually good to mimic the English word order:

necessary not = $\Box\sim$

not necessary = $\sim\Box$

necessary if = $\Box($

if necessary = $(\Box$

Each “necessary” or “possible” uses a separate box or diamond:

If A is necessary and B is possible, then C is possible = $((\Box A \cdot \Diamond B) \supset \Diamond C)$

This ambiguous sentence could
have either of two meanings:

“If you’re a bachelor, then you must be unmarried.”

Simple Necessity

$$(B \supset \Box U)$$

If you’re a bachelor, then
you’re *inherently unmarriageable*
(in no possible world would
anyone ever marry you).

If B, then U (by itself) is necessary.

Conditional Necessity

$$\Box(B \supset U)$$

It’s necessary that
if you’re a bachelor
then you’re unmarried.

It’s necessary that if-B-then-U.

These forms aren't ambiguous:

If A, then B (by itself) is necessary	=	$(A \supset \Box B)$
If A, then B is inherently necessary	=	$(A \supset \Box B)$
A entails B	=	$\Box(A \supset B)$
Necessarily, if A then B	=	$\Box(A \supset B)$
It's necessary that if A then B	=	$\Box(A \supset B)$
“If A then B” is a necessary truth	=	$\Box(A \supset B)$

These forms are ambiguous:

*“If A, then it's necessary
(must be) that B”*
could mean $(A \supset \Box B)$
or $\Box(A \supset B)$.

*“If A, then it's impossible
(couldn't be) that B”*
could mean $(A \supset \Box \sim B)$
or $\Box(A \supset \sim B)$.

A is possible (could be true) = $\diamond A$
 A is necessary (must be true) = $\square A$
 A is impossible (self-contradictory) = $\sim \diamond A = \square \sim A$

A is consistent with B = $\diamond(A \cdot B)$
 A entails B = $\square(A \supset B)$

A is a contingent statement = $(\diamond A \cdot \diamond \sim A)$
 A is a contingent truth = $(A \cdot \diamond \sim A)$

If A, then it's necessary that B = $(A \supset \square B)$ or $\square(A \supset B)$
 If A, then it's impossible that B = $(A \supset \square \sim B)$ or $\square(A \supset \sim B)$

A *world prefix* is a string of zero or more instances of “W.”

A *derived step* is now a line consisting of a world prefix and then “∴” and then a wff.

∴ A	(So A is true in the actual world.)
-----	--

W ∴ A	(So A is true in world W.)
-------	-------------------------------

An *assumption* is now a line consisting of a world prefix and then “asm:” and then a wff.

asm: A	(Assume A is true in the actual world.)
--------	---

W asm: A	(Assume A is true in world W.)
----------	--------------------------------------

Modal Inference Rules

First reverse
squiggles

$$\sim\Box A \rightarrow \Diamond\sim A$$

$$\sim\Diamond A \rightarrow \Box\sim A$$

*

and drop
diamonds;

$$\Diamond A \rightarrow W :. A,$$

use a *new* string of W's

*

lastly, drop
boxes.

$$\Box A \rightarrow W :. A,$$

use any world prefix

Don't
star

Valid

* 1 $\diamond(A \cdot B)$

[$\therefore \diamond A$

* 2 asm: $\sim \diamond A$

3 $\therefore \Box \sim A$ {from 2}

← reverse squiggles

* 4 $W \therefore (A \cdot B)$ {from 1}

← drop diamonds

5 $W \therefore A$ {from 4}

6 $W \therefore B$ {from 4}

7 $W \therefore \sim A$ {from 3}

← drop boxes

8 $\therefore \diamond A$ {from 2; 5 contradicts 7}

1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world.
(Never use a new world when you drop a box.)

Drop
boxes

$\Box A \rightarrow W \therefore A,$ use any world prefix
--

Drop a box into all worlds with W's.

Drop a box into the actual world just if:

- you have an unmodalized instance of a letter in your original premises or conclusion, or
- you've done everything else possible (including further assumptions if needed) and still have no other worlds.

- * 1 $\diamond \sim H$
- 2 $\Box(H \vee T)$
- [$\therefore \Box T$
- * 3 asm: $\sim \Box T$
- * 4 $\therefore \diamond \sim T$ {from 3}
- 5 $W \therefore \sim T$ {from 4}
- 6 $WW \therefore \sim H$ {from 1}
- * 7 $W \therefore (H \vee T)$ {from 2}
- * 8 $WW \therefore (H \vee T)$ {from 2}
- 9 $W \therefore H$ {from 5 and 7}
- 10 $WW \therefore T$ {from 6 and 8}

Invalid

W	H, $\sim T$
WW	T, $\sim H$

If you can't get a contradiction, construct a refutation.

$$\begin{aligned}
 1 \quad & \diamond \sim H = 1 \\
 2 \quad & \Box(H \vee T) = 1 \\
 & [\therefore \Box T = 0]
 \end{aligned}$$

Invalid

W	H, $\sim T$
WW	T, $\sim H$

“ $\diamond A$ ” is true if and only if *at least one world* has “A” true.

“ $\Box A$ ” is true if and only if *all worlds* have “A” true.

If a wff doesn't start with a box or diamond: evaluate each subformula that starts with a box or diamond, and then substitute “1” or “0” for it:

$\sim \Box H$	$(\diamond H \supset \Box T)$	$\sim \Box(H \vee T)$
$= \sim 0$	$= (1 \supset 0)$	$= \sim 1$
$= 1$	$= 0$	$= 0$

* 1 $\diamond(A \cdot B)$ Valid
 [$\therefore \diamond A$
 * 2 asm: $\sim \diamond A$
 3 $\therefore \Box \sim A$ {from 2}
 * 4 $W \therefore (A \cdot B)$ {from 1}
 5 $W \therefore A$ {from 4}
 6 $W \therefore B$ {from 4}
 7 $W \therefore \sim A$ {from 3}
 8 $\therefore \diamond A$ {from 2; 5 contradicts 7}

H, $\sim T$
T, $\sim H$

* 1 $\diamond \sim H$ W
 2 $\Box(H \vee T)$ WW
 [$\therefore \Box T$
 * 3 asm: $\sim \Box T$
 * 4 $\therefore \diamond \sim T$ {from 3} Invalid
 5 $W \therefore \sim T$ {from 4}
 6 $WW \therefore \sim H$ {from 1}
 * 7 $W \therefore (H \vee T)$ {from 2}
 * 8 $WW \therefore (H \vee T)$ {from 2}
 9 $W \therefore H$ {from 5 and 7}
 10 $WW \therefore T$ {from 6 and 8}

1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world.
 (Never use a new world when you drop a box.)

Do these two now:

$$\begin{array}{ll} (\diamond A \supset \diamond B) & (A \cdot \diamond B) \\ \therefore \Box(A \supset B) & \Box((A \cdot B) \supset C) \\ & \therefore C \end{array}$$

1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world.
(Never use a new world when you drop a box.)

If you get stuck, make another assumption by breaking up a wff of the form “ $\sim(A \cdot B)$,” “ $(A \vee B)$,” or “ $(A \supset B)$.”