

Propositional logic

$\sim P$ = Not-P

$(P \cdot Q)$ = Both P and Q

$(P \vee Q)$ = Either P or Q

$(P \supset Q)$ = If P then Q

$(P \equiv Q)$ = P if and only if Q

1. Any capital letter is a wff.
2. The result of prefixing any wff with “ \sim ” is a wff.
3. The result of joining any two wffs by “ \cdot ” or “ \vee ” or “ \supset ” or “ \equiv ” and enclosing the result in parentheses is a wff.

Two translation rules

Put “(” wherever you see “both,” “either,” or “if.”

Either not A or B = $(\sim A \vee B)$

Not either A or B = $\sim(A \vee B)$

Group together parts on either side of a comma.

If A, then B and C = $(A \supset (B \cdot C))$

If A then B, and C = $((A \supset B) \cdot C)$

Propositional translations

$\sim P$ = Not-P

$(P \cdot Q)$ = Both P and Q

$(P \vee Q)$ = Either P or Q

$(P \supset Q)$ = If P then Q

$(P \equiv Q)$ = P if and only if Q

Put “(” wherever
you see “both,”
“either,” or “if.”

Group together
parts on either
side of a comma.

“I went to Paris *and* I went to Quebec.”

P	Q	(P · Q)	
0	0	0	(0 · 0) = 0
0	1	0	(0 · 1) = 0
1	0	0	(1 · 0) = 0
1	1	1	(1 · 1) = 1

“(P · Q)” claims that *both* parts are true.

“(P · Q)” is a *conjunction*; P and Q are its *conjuncts*.

“I went to Paris *or* I went to Quebec.”

P	Q	$(P \vee Q)$	
0	0	0	$(0 \vee 0) = 0$
0	1	1	$(0 \vee 1) = 1$
1	0	1	$(1 \vee 0) = 1$
1	1	1	$(1 \vee 1) = 1$

“(P \vee Q)” claims that *at least one* part is true.

“(P \vee Q)” is a *disjunction*; P and Q are its *disjuncts*.

“If I went to Paris, *then* I went to Quebec.”

P	Q	$(P \supset Q)$	
0	0	1	$(0 \supset 0) = 1$
0	1	1	$(0 \supset 1) = 1$
1	0	0	$(1 \supset 0) = 0$
1	1	1	$(1 \supset 1) = 1$

“(P \supset Q)” says we *don't* have the first part true and the second false.
“(P \supset Q)” is a *conditional*; P is the *antecedent* and Q the *consequent*.

Falsity implies anything.	$(0 \supset \) = 1$
Anything implies truth.	$(\supset 1) = 1$
Truth doesn't imply falsity.	$(1 \supset 0) = 0$

P	Q	$(P \equiv Q)$		
0	0	1	$(0 \equiv 0) = 1$	“I went to Paris, <i>if and only if</i> I went to Quebec.”
0	1	0	$(0 \equiv 1) = 0$	
1	0	0	$(1 \equiv 0) = 0$	
1	1	1	$(1 \equiv 1) = 1$	

“(P \equiv Q)” claims that both parts have the *same* truth value.

“(P \equiv Q)” is a *biconditional*.

P	$\sim P$		
0	1	$\sim 0 = 1$	“I <i>didn't</i> go to Paris.”
1	0	$\sim 1 = 0$	

“ $\sim P$ ” has the *opposite* value of “P.”

“ $\sim P$ ” is a *negation*.

Basic Truth Equivalences

AND	OR	IF-THEN	IFF	NOT
$(0 \cdot 0) = 0$	$(0 \vee 0) = 0$	$(0 \supset 0) = 1$	$(0 \equiv 0) = 1$	
$(0 \cdot 1) = 0$	$(0 \vee 1) = 1$	$(0 \supset 1) = 1$	$(0 \equiv 1) = 0$	$\sim 0 = 1$
$(1 \cdot 0) = 0$	$(1 \vee 0) = 1$	$(1 \supset 0) = 0$	$(1 \equiv 0) = 0$	$\sim 1 = 0$
$(1 \cdot 1) = 1$	$(1 \vee 1) = 1$	$(1 \supset 1) = 1$	$(1 \equiv 1) = 1$	

*both parts
are true*

*at least one
part is true*

*we don't have first
true & second false*

*both parts have
same truth value*

*reverse the
truth value*

Falsity implies anything.
Anything implies truth.
Truth doesn't imply falsity.

Truth Evaluations

Assume that $A=1$ and $X=0$.

What is the truth value of “ $\sim(A \cdot X)$ ”?

- Plug in “1” and “0” for the letters. $\sim(A \cdot X)$
- Simplify from the inside out, until you get “1” or “0.” $\sim(1 \cdot 0)$
 ~ 0
1

Unknown Evaluations

Assume that $T=1$ and $F=0$ and $U=?$ (unknown).

What is the truth value of “ $(U \cdot F)$ ”?

“ $(? \cdot 0)$ ” has to be 0 because:

An AND with
one part 0 is 0.

It comes out 0 both ways:
 $(1 \cdot 0) = 0$ and $(0 \cdot 0) = 0$.

Complex Truth Tables: do one for “ $(P \equiv \sim Q)$ ”

P	Q	$(P \equiv \sim Q)$	
0	0	0	$(0 \equiv \sim 0) = (0 \equiv 1) = 0$
0	1	1	$(0 \equiv \sim 1) = (0 \equiv 0) = 1$
1	0	1	$(1 \equiv \sim 0) = (1 \equiv 1) = 1$
1	1	0	$(1 \equiv \sim 1) = (1 \equiv 0) = 0$

- Write the formula: “ $(P \equiv \sim Q)$ ”
- On the left, write each letter in the formula: “P” and “Q”
- Below this, write each truth combination; n letters give 2^n truth combinations.
- Figure out the value of the formula on each combination.

VALID = No possible case has premises all true and conclusion false.

1
1
$\therefore 0$

INVALID = Some possible case has premises all true and conclusion false.

1
1
 $\therefore 0$

Truth-table test: Construct a truth table showing the truth value of the premises and conclusion for all possible cases. The argument is **VALID** if and only if no possible case has premises all true and conclusion false.

C	D	$(C \supset D),$	D	\therefore	C
0	0	1	0	0	0
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

← **Invalid**

On the short-cut truth-table test, evaluate easier wffs first and cross out rows that couldn't have true premises and a false conclusion.

C	D	$(C \supset D),$	D	\therefore	C
0	0	----- 0 -----	0	-----	-----
0	1		1		
1	0	----- 0 -----	0	-----	-----
1	1		1		

First do “D” – and cross out rows that couldn't be 110.

C	D	$(C \supset D),$	D	\therefore	C
0	0	-----	0	-----	-----
0	1		1		0
1	0	-----	0	-----	-----
1	1	-----	1	-----	1

Then do
 “C” – and
 cross out
 rows that
 couldn’t
 be 110.

C	D	$(C \supset D),$	D	\therefore	C
0	0	-----	0	-----	-----
0	1	1	1		0
1	0	-----	0	-----	-----
1	1	-----	1	-----	1

Finally, do
 “ $(C \supset D).$ ”

← **Invalid**

VALID = No possible case has premises all true and conclusion false.

1
1
∴ 0

INVALID = Some possible case has premises all true and conclusion false.

1
1
∴ 0

Truth-assignment test: Set each premise to 1 and the conclusion to 0. Figure out the truth value of as many letters as possible. The argument is **VALID** if and only if no possible way to assign 1 and 0 to the letters will keep the premises all 1 and conclusion 0.

$$\begin{array}{l}
 (L \vee R) = 1 \\
 \sim L = 1 \\
 \therefore R = 0
 \end{array}
 \rightarrow
 \begin{array}{l}
 (L^0 \vee R^0) = 1 \\
 \sim L^0 = 1 \\
 \therefore R^0 = 0
 \end{array}
 \rightarrow
 \begin{array}{l}
 \frac{0}{(L^0 \vee R^0) \neq 1} \text{ Valid} \\
 \sim L^0 = 1 \\
 \therefore R^0 = 0
 \end{array}$$

Harder Translations

Translate “but” (“yet,”
“however,” “although,”
and so on) as “and.”

Northwestern played,
but it won
= $(N \cdot W)$

Translate “unless”
as “or.”

You’ll die *unless* you
give me your money
= $(D \vee M)$

Translate “just if” and
“iff” (a logician word)
as “if and only if.”

I’ll take the job *just if*
you pay me a million
= $(J \equiv M)$

The part after “if” (“provided that,” “assuming that,” and so on) is the if-part (the antecedent, the part before the horseshoe).

Gensler is happy *if* Michigan wins.
= *If* Michigan wins, Gensler is happy.
= $(M \supset H)$

The part after “only if” is the then-part (the consequent, the part after the horseshoe). (Or write “ \supset ” for “only if.”)

You’re alive *only if* you have oxygen.
= *Only if* you have oxygen are you alive.
= $(A \supset O) = (\sim O \supset \sim A)$

Oxygen is *sufficient* for life = $(O \supset L)$

Oxygen is *necessary* for life = $(\sim O \supset \sim L)$

Oxygen is *necessary and sufficient* for life = $(O \equiv L)$

Harder Translations

- BUT = YET = HOWEVER = ALTHOUGH = AND.
- UNLESS = OR.
- JUST IF = IFF = IF AND ONLY IF.
- The part after “if” (“provided that,” “assuming that,” and so on) goes *before* the horseshoe.
- The part after “only if” goes *after* the horseshoe.
- Oxygen is *sufficient for* life = $(O \supset L)$
- Oxygen is *necessary for* life = $(\sim O \supset \sim L)$
- Oxygen is *necessary and sufficient for* life = $(O \equiv L)$

Idiomatic arguments

1. Identify the conclusion.

These often indicate premises:

Because, for, since, after all ...

I assume that, as we know ...

For these reasons ...

-
-

These often indicate conclusions:

Hence, thus, so, therefore ...

It must be, it can't be ...

This proves (or shows) that ...

2. Translate into logic, using wffs. Add implicit premises if needed.

3. Test for validity.

S-Rules

AND

This and that.	$(P \cdot Q)$	AND statement, so both parts are true.
\therefore This.	$\frac{\quad}{P, Q}$	
\therefore That.		

NOR

Not either this or that.	$\sim(P \vee Q)$	NOT-EITHER is true, so both are false.
\therefore Not this.	$\frac{\quad}{\sim P, \sim Q}$	
\therefore Not that.		

NIF

False if-then.	$\sim(P \supset Q)$	FALSE IF-THEN, so first part true, second part false.
\therefore First part true.	$\frac{\quad}{P, \sim Q}$	
\therefore Second part false.		

S-Rules: When can we simplify?

CAN SIMPLIFY

AND	$(P \cdot Q) \rightarrow P, Q$
NOR	$\sim(P \vee Q) \rightarrow \sim P, \sim Q$
NIF	$\sim(P \supset Q) \rightarrow P, \sim Q$

CAN'T

$\sim(P \cdot Q)$
$(P \vee Q)$
$(P \supset Q)$

Recall our basic truth tables:

For an AND to be true, both parts have to be true.

For an OR to be false, both parts have to be false.

For an IF-THEN to be false, we need part 1 true and part 2 false.

I-Rules

Not both are true. This one is true. ∴ The other isn't.	$\frac{\sim(P \cdot Q)}{P}$ $\sim Q$	$\frac{\sim(P \cdot Q)}{Q}$ $\sim P$	<i>Deny AND.</i> <i>Affirm one part.</i> ∴ <i>Deny other part.</i>
---	---	---	--

At least one is true. This one isn't. ∴ The other is.	$\frac{(P \vee Q)}{\sim P}$ Q	$\frac{(P \vee Q)}{\sim Q}$ P	<i>Affirm OR.</i> <i>Deny one part.</i> ∴ <i>Affirm other part.</i>
---	------------------------------------	------------------------------------	---

IF-THEN. Affirm first. ∴ Affirm second.	$\frac{(P \supset Q)}{P}$ Q
---	----------------------------------

IF-THEN. Deny second. ∴ Deny first.	$\frac{(P \supset Q)}{\sim Q}$ $\sim P$
---	--

I-Rules: When can we infer?

With NAND or OR, premises must *alternate* affirming and denying.

<p><i>Deny</i> AND. <i>Affirm</i> one part. \therefore <i>Deny</i> other part.</p>	<p>NAND</p>	<p>$\sim(P \cdot Q), P \rightarrow \sim Q$ $\sim(P \cdot Q), Q \rightarrow \sim P$</p>
<p><i>Affirm</i> OR. <i>Deny</i> one part. \therefore <i>Affirm</i> other part.</p>	<p>OR</p>	<p>$(P \vee Q), \sim P \rightarrow Q$ $(P \vee Q), \sim Q \rightarrow P$</p>

With IF-THEN, we need part 1 true
 or part 2 false: “(+ \supset -).”

<p>$(P \supset Q), P \rightarrow Q$ $(P \supset Q), \sim Q \rightarrow \sim P$</p>

S- and I-Rules

S-rules

$$(P \cdot Q) \rightarrow P, Q$$
$$\sim(P \vee Q) \rightarrow \sim P, \sim Q$$
$$\sim(P \supset Q) \rightarrow P, \sim Q$$

I-rules

$$\sim(P \cdot Q), P \rightarrow \sim Q$$
$$\sim(P \cdot Q), Q \rightarrow \sim P$$
$$(P \vee Q), \sim P \rightarrow Q$$
$$(P \vee Q), \sim Q \rightarrow P$$
$$(P \supset Q), P \rightarrow Q$$
$$(P \supset Q), \sim Q \rightarrow \sim P$$

Extended S- and I-rule inferences

$$\frac{\sim((A \cdot B) \supset \sim C)}{(A \cdot B), C}$$

FALSE IF-THEN,
so first part true,
second part false.

$$\frac{\sim(\\$\\$\\$ \supset \\###)}{\\$\\$, \sim\\###}$$

$$\frac{((A \cdot B) \supset \sim C) \quad \sim(A \cdot B)}{\text{nil}}$$

IF-THEN:
need first part true
or second part false

$$\frac{(\\$\\$\\$ \supset \\###) \quad \sim\\$\\$}{\text{nil}}$$

Logic gates and computers



A	B	$(A \cdot B)$
0	0	0
0	1	0
1	0	0
1	1	1

An AND-GATE
gives an output
voltage if and only
if both inputs have
an input voltage.