

$(\sim B \supset (A \cdot C))$   
 $(A \supset \sim C)$   
 $\therefore B$

This is our  
sample  
argument.

# Formal Proofs

From now on, formal proofs will be our main way to test arguments. We'll begin with easier proofs. Our initial strategy for constructing proofs has three steps.

1  $(\sim B \supset (A \cdot C))$

2  $(A \supset \sim C)$

[  $\therefore B$

3 asm:  $\sim B$

[ Block off conclusion

← Assume the opposite

## Step 1: START

Block off the conclusion and add “asm:” followed by the conclusion’s simpler contradictory.

<p>* 1    <math>(\sim B \supset (A \cdot C))</math>  2    <math>(A \supset \sim C)</math>     [ <math>\therefore B</math>  3    asm: <math>\sim B</math>  4    <math>\therefore (A \cdot C)</math>    {from 1 and 3}    ←</p>	<p>I-rule inference:</p> $\frac{(\sim B \supset (A \cdot C)) \quad \sim B}{(A \cdot C)}$
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## Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using the S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

- \* 1  $(\sim B \supset (A \cdot C))$
- 2  $(A \supset \sim C)$
- [  $\therefore B$
- 3 asm:  $\sim B$
- \* 4  $\therefore (A \cdot C)$  {from 1 and 3}
- 5  $\therefore A$  {from 4}
- 6  $\therefore C$  {from 4}

S-rule inference:

$$\frac{(A \cdot C)}{A, C}$$

←

←

## Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using the S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

- \* 1  $(\sim B \supset (A \cdot C))$
- \* 2  $(A \supset \sim C)$
- [  $\therefore B$
- 3 asm:  $\sim B$
- \* 4  $\therefore (A \cdot C)$  {from 1 and 3}
- 5  $\therefore A$  {from 4}
- 6  $\therefore C$  {from 4}
- 7  $\therefore \sim C$  {from 2 and 5}

I-rule inference:

$$\frac{(A \supset \sim C) \quad A}{\sim C}$$



## Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using the S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

<p>* 1    <math>(\sim B \supset (A \cdot C))</math></p> <p>* 2    <math>(A \supset \sim C)</math></p> <p>   [ <math>\therefore B</math></p> <p>   3    [ asm: <math>\sim B</math></p> <p>* 4    [ <math>\therefore (A \cdot C)</math> {from 1 and 3}</p> <p>   5    [ <math>\therefore A</math> {from 4}</p> <p>   6    [ <math>\therefore C</math> {from 4}</p> <p>   7    [ <math>\therefore \sim C</math> {from 2 and 5}</p> <p>   8 <math>\therefore B</math> {from 3; 6 contradicts 7} ←</p>	<p>Valid</p> <p>block off from the assumption on down</p> <p>derive conclusion</p>
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### Step 3: RAA

When some pair of not-blocked-off lines contradicts, apply RAA and derive the original conclusion. Your proof is done!

### *S-rules*

$$(P \cdot Q) \rightarrow P, Q$$

$$\sim(P \vee Q) \rightarrow \sim P, \sim Q$$

$$\sim(P \supset Q) \rightarrow P, \sim Q$$

$$\sim\sim P \rightarrow P$$

$$(P \equiv Q) \rightarrow (P \supset Q), (Q \supset P)$$

$$\sim(P \equiv Q) \rightarrow (P \vee Q), \sim(P \cdot Q)$$

### *I-rules*

$$\sim(P \cdot Q), P \rightarrow \sim Q$$

$$\sim(P \cdot Q), Q \rightarrow \sim P$$

$$(P \vee Q), \sim P \rightarrow Q$$

$$(P \vee Q), \sim Q \rightarrow P$$

$$(P \supset Q), P \rightarrow Q$$

$$(P \supset Q), \sim Q \rightarrow \sim P$$

*RAA*: Suppose that some pair of not-blocked-off lines has contradictory wffs. Then block off all the lines from the last not-blocked-off assumption on down and infer a line consisting in “ $\therefore$ ” followed by a contradictory of that assumption.

*	1	$(\sim B \supset (A \cdot C))$	←	premises (no “asm” or “∴”)
*	2	$(A \supset \sim C)$	←	
		[ ∴ B		
	3	asm: $\sim B$	←	assumption (“asm”)
*	4	$\therefore (A \cdot C)$ {from 1 and 3}	←	derived lines (“∴”)
	5	$\therefore A$ {from 4}	←	
	6	$\therefore C$ {from 4}	←	
	7	$\therefore \sim C$ {from 2 and 5}	←	
	8	$\therefore B$ {from 3; 6 contradicts 7}	←	

A *formal proof* is a vertical sequence of zero or more premises followed by one or more assumptions or derived line, where each derived line follows from previously not-blocked-off lines by RAA or one of the inference rules, and each assumption is blocked off using RAA.

Two wffs are *contradictory* if they are exactly alike except that one starts with an additional “ $\sim$ .”

A *simple wff* is a letter or its negation; any other wff is *complex*.

Valid

- \* 1  $(\sim B \supset (A \cdot C))$
- \* 2  $(A \supset \sim C)$
- [  $\therefore B$
- 3 [ asm:  $\sim B$
- \* 4 [  $\therefore (A \cdot C)$  {from 1 and 3}
- 5 [  $\therefore A$  {from 4}
- 6 [  $\therefore C$  {from 4}
- 7 [  $\therefore \sim C$  {from 2 and 5}
- 8  $\therefore B$  {from 3; 6 contradicts 7}

## Proof Strategy

- 1 START: Assume the opposite of the conclusion.
- 2 S&I: Derive whatever you can using the S- and I-rules, until you get a contradiction.
- 3 RAA: Apply RAA and derive the original conclusion.

- |   |                             |  |
|---|-----------------------------|--|
| 1 | $(A \supset B)$             |  |
|   | $[\therefore (B \supset A)$ |  |
| * | 2                           | asm: $\sim(B \supset A)$                           |
|   | 3                           | $\therefore B$ {from 2}      We can derive         |
|   | 4                           | $\therefore \sim A$ {from 2}      nothing further. |

## Proof strategy to include invalid arguments:

- 1 **START:** Assume the opposite of the conclusion.
- 2 **S&I:** Derive whatever you can using the S- and I-rules.
- 3 **RAA:** If you get a contradiction, apply RAA and derive the original conclusion.
- 4 **REFUTE:** If you don't get a contradiction, construct a refutation box.

- 1  $(A^0 \supset B^1) = 1$   
 $[\therefore (B^1 \supset A^0) = 0$
- \* 2 asm:  $\sim(B \supset A)$
- 3  $\therefore B$  {from 2}
- 4  $\therefore \sim A$  {from 2}

Invalid

$B, \sim A$
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Step 4 – REFUTE: If you can't get a contradiction, then:

- draw a box containing any simple wffs (letters or their negation) that aren't blocked off;
- in the original argument, mark each letter "1" or "0" or "?" depending on whether you have the letter or its negation or neither in the box;
- if these truth conditions make the premises all true and conclusion false, then this shows the argument to be invalid.

\* 1  $(\sim B \supset (A \cdot C))$  Valid

\* 2  $(A \supset \sim C)$

[  $\therefore B$

3 asm:  $\sim B$

\* 4  $\therefore (A \cdot C)$  {from 1 and 3}

5  $\therefore A$  {from 4}

6  $\therefore C$  {from 4}

7  $\therefore \sim C$  {from 2 and 5}

8  $\therefore B$  {from 3; 6 contradicts 7}

1  $(A^0 \supset B^1) = 1$  Invalid

[  $\therefore (B^1 \supset A^0) = 0$

\* 2 asm:  $\sim(B \supset A)$  B,  $\sim A$

3  $\therefore B$  {from 2}

4  $\therefore \sim A$  {from 2}

- 1 START: Assume the opposite of the conclusion.
- 2 S&I: Derive whatever you can using the S- and I-rules.
- 3 RAA: If you get a contradiction, apply RAA and derive the original conclusion.
- 4 REFUTE: If you don't get a contradiction, construct a refutation box.

1  $(B \vee A)$

2  $(B \supset A)$

[  $\therefore \sim(A \supset \sim A)$

3 asm:  $(A \supset \sim A)$



We're stuck!

*We get stuck using our old strategy – so we need to make another assumption.*

- 1 START: Assume the opposite of the conclusion.
- 2 S&I: Derive whatever you can using the S- and I-rules.
- 3 RAA: If you get a contradiction, apply RAA and derive the original conclusion.
- 4 REFUTE: If you don't get a contradiction, construct a refutation box.

1  $(B \vee A)$

2  $(B \supset A)$

[  $\therefore \sim(A \supset \sim A)$

3 asm:  $(A \supset \sim A)$



We're stuck!

We're stuck when:

We can't apply the S- or I-rules further.

And we can't prove the argument VALID  
(since we don't have a contradiction)  
or INVALID (since we don't have enough  
simple wffs for a refutation).

1  $(B \vee A)$

2  $(B \supset A)$

[ $\therefore \sim(A \supset \sim A)$

3 asm:  $(A \supset \sim A)$

4 asm: B {break up 1}

When you're stuck,  
try to make another  
assumption.



ASSUME: Look for a complex wff that isn't starred or blocked off or broken. This wff will have one of these forms:

$\sim(A \cdot B)$        $(A \vee B)$        $(A \supset B)$

Assume one side or its negation – and then return to step 2 (S&I).

- 1     $(B \vee A)$
  - \*\* 2     $(B \supset A)$
  - $[\therefore \sim(A \supset \sim A)]$
  - \*\* 3    asm:  $(A \supset \sim A)$
  - 4    asm: B    {break up 1}
  - 5     $\therefore A$     {from 2 and 4}                    ←
  - 6     $\therefore \sim A$     {from 3 and 5}                   ←
- Contradiction!

S&I: Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using the S- and I-rules. Star (*with one star for each live assumption*) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

1  $(B \vee A)$   
 2  $(B \supset A)$   
 [  $\therefore \sim(A \supset \sim A)$   
 3 asm:  $(A \supset \sim A)$   
 4 [ asm: B {break up 1}  
 5 [  $\therefore A$  {from 2 and 4}  
 6 [  $\therefore \sim A$  {from 3 and 5}  
 7  $\therefore \sim B$  {from 4; 5 contradicts 6}      ←      Apply RAA.

**RAA:** If you have a contradiction, apply RAA on the last live assumption. If all assumptions are now blocked off, you've proved the argument valid. *Otherwise, erase star strings having more stars than the number of live assumptions* – and then return to step 2 (S&I).

# Valid

- \* 1  $(B \vee A)$
- 2  $(B \supset A)$
- [  $\therefore \sim(A \supset \sim A)$
- \* 3  $\text{asm: } (A \supset \sim A)$
- 4  $\left[ \begin{array}{l} \text{asm: } B \quad \{\text{break up 1}\} \\ \therefore A \quad \{\text{from 2 and 4}\} \\ \therefore \sim A \quad \{\text{from 3 and 5}\} \end{array} \right.$
- 7  $\therefore \sim B \quad \{\text{from 4; 5 contradicts 6}\}$
- 8  $\therefore A \quad \{\text{from 1 and 7}\}$  ←
- 9  $\therefore \sim A \quad \{\text{from 3 and 8}\}$  ←
- 10  $\therefore \sim(A \supset \sim A) \quad \{\text{from 3; 8 contradicts 9}\}$  ←

We use “ $\sim B$ ”  
to get a  
contradiction &  
finish the proof.

- \* 1  $(B \vee A)$  Valid
- 2  $(B \supset A)$
- [  $\therefore \sim(A \supset \sim A)$
- \* 3 [ asm:  $(A \supset \sim A)$
- 4 [ [ asm: B {break up 1}
- 5 [ [  $\therefore A$  {from 2 and 4}
- 6 [ [  $\therefore \sim A$  {from 3 and 5}
- 7 [  $\therefore \sim B$  {from 4; 5 contradicts 6}
- 8 [  $\therefore A$  {from 1 and 7}
- 9 [  $\therefore \sim A$  {from 3 and 8}
- 10  $\therefore \sim(A \supset \sim A)$  {from 3; 8 contradicts 9}

Strategy:

Start
S&I
RAA
Assume
Refute

- 1  $\sim(A \cdot B)$   
[  $\therefore (\sim A \cdot \sim B)$   
2 asm:  $\sim(\sim A \cdot \sim B)$  ← Assume opposite.

Then we're stuck!

We can't apply the S- or I-rules or RAA; and we don't have enough simple wffs for a refutation.

START: Assume the opposite of the conclusion.

- 1  $\sim(A \cdot B)$
- [  $\therefore (\sim A \cdot \sim B)$
- 2 asm:  $\sim(\sim A \cdot \sim B)$
- 3 asm: A {break up 1} ←

When you're stuck,  
try to make another  
assumption.

**ASSUME:** Look for a complex wff that isn't starred or blocked off or broken. This wff will have one of these forms:

$$\sim(A \cdot B) \quad (A \vee B) \quad (A \supset B)$$

Assume one side or its negation – and then return to step 2 (S&I).

- \*\* 1     $\sim(A \cdot B)$   
       [  $\therefore (\sim A \cdot \sim B)$   
 2    asm:  $\sim(\sim A \cdot \sim B)$   
 3    asm: A    {break up 1}  
 4     $\therefore \sim B$     {from 1 and 3}    ←    Derive further lines.

We're stuck again! But now all complex wffs are either starred or blocked off or broken.

S&I: Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using the S- and I-rules. Star (*with one star for each live assumption*) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

\*\* 1  $\sim(A^1 \cdot B^0) = 1$   
 [  $\therefore (\sim A^1 \cdot \sim B^0) = 0$   
 2 asm:  $\sim(\sim A \cdot \sim B)$   
 3 asm: A {break up 1}  
 4  $\therefore \sim B$  {from 1 and 3}

Invalid

A, $\sim B$
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<p>REFUTE: Construct a refutation box if you can't apply S- and I-rules or RAA further, and yet all complex wffs are either starred or blocked off or broken.</p>
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Do these two now:

$(A \supset (B \cdot C))$

$(\sim A \supset (D \vee E))$

$(B \supset \sim C)$

$\sim D$

$\therefore E$

$((A \vee B) \supset (C \supset D))$

$(B \supset C)$

$(E \supset (F \vee B))$

$\therefore (A \supset B)$

Start	S&I	RAA	Assume	Refute
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\* 1 (B  $\vee$  A) Valid

2 (B  $\supset$  A)

[  $\therefore \sim(A \supset \sim A)$

\* 3 asm: (A  $\supset$   $\sim A$ )

4 [ asm: B {break up 1}

5 [  $\therefore A$  {from 2 and 4}

6 [  $\therefore \sim A$  {from 3 and 5}

7  $\therefore \sim B$  {from 4; 5 contradicts 6}

8  $\therefore A$  {from 1 and 7}

9  $\therefore \sim A$  {from 3 and 8}

10  $\therefore \sim(A \supset \sim A)$  {from 3; 8 contradicts 9}

\*\* 1  $\sim(A^1 \cdot B^0) = 1$  Invalid

[  $\therefore (\sim A^1 \cdot \sim B^0) = 0$

2 asm:  $\sim(\sim A \cdot \sim B)$

3 asm: A {break up 1}

4  $\therefore \sim B$  {from 1 and 3}

A, $\sim B$
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Start	S&I	RAA	Assume	Refute
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Standard inferential proofs use RAA & conditional proofs, plus these:

Inference  
rules

$$\begin{array}{ll}
 (P \cdot Q) \rightarrow P & (P \supset Q), P \rightarrow Q \\
 P, Q \rightarrow (P \cdot Q) & (P \supset Q), \sim Q \rightarrow \sim P \\
 (P \vee Q), \sim P \rightarrow Q & (P \supset Q), (Q \supset R) \rightarrow (P \supset R) \\
 P \rightarrow (P \vee Q) & (P \supset Q) \rightarrow (P \supset (P \cdot Q)) \\
 & ((P \supset Q) \cdot (R \supset S)), (P \vee R) \rightarrow (Q \vee S)
 \end{array}$$

Equivalence  
rules

$$\begin{array}{ll}
 P \equiv \sim\sim P & (P \supset Q) \equiv (\sim P \vee Q) \\
 P \equiv (P \cdot P) & (P \cdot (Q \cdot R)) \equiv ((P \cdot Q) \cdot R) \\
 P \equiv (P \vee P) & (P \vee (Q \vee R)) \equiv ((P \vee Q) \vee R) \\
 (P \cdot Q) \equiv (Q \cdot P) & (P \cdot (Q \vee R)) \equiv ((P \cdot Q) \vee (P \cdot R)) \\
 (P \vee Q) \equiv (Q \vee P) & (P \vee (Q \cdot R)) \equiv ((P \vee Q) \cdot (P \vee R)) \\
 \sim(P \cdot Q) \equiv (\sim P \vee \sim Q) & (P \equiv Q) \equiv ((P \supset Q) \cdot (Q \supset P)) \\
 \sim(P \vee Q) \equiv (\sim P \cdot \sim Q) & (P \equiv Q) \equiv ((P \cdot Q) \vee (\sim P \cdot \sim Q)) \\
 (P \supset Q) \equiv (\sim Q \supset \sim P) & ((P \cdot Q) \supset R) \equiv (P \supset (Q \supset R))
 \end{array}$$