Dislocations and the nematic—to—smectic-A transition for arbitrary values of K_1

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The de Gennes model is used to derive the energy of interacting dislocations in smectic-A liquid crystals. This energy reduces to the energy of interacting vortices in type-II superconductors when the splay elastic constant K_1 is zero and to that derived from the Landau-Peierls elastic energy when spatial variations are slow on a scale of the bend and twist penetration depths. Furthermore, it has a well-defined $K_1 \to \infty$ limit. The dislocation energy is used to study the nematic—to—smectic-A transition as a function of K_1 in two dimensions and in $4-\epsilon$ dimensions. No evidence for the Nelson-Toner, anisotropic critical point is found in either case.

I. INTRODUCTION

Topological excitations such as vortices and dislocations are known to play an important role in two-dimensional systems.[1] In particular, they are responsible for the Kosterlitz-Thouless transitions in superfluids,[2] for dislocation melting and hexatic phases[3] in two-dimensional solids, and for similar phenomena in a variety of other systems. In addition, they lead to quasi—Kosterlitz-Thouless transitions in dirty superconducting[4] films with linear dimensions L much less than the effective penetration depth λ. In cleaner superconductors and in two-dimensional smectics, topological excitations are expected to wash out phase transitions altogether.

Recently, the role of topological excitations in three-dimensional systems, particularly smectic liquid crystals[5–7] and superconductors,[8] has received greater attention. Helfrich[5] has argued that if the energy per unit length ϵ of a dislocation line is finite, then both the energy and entropy of a single large dislocation loop will be proportional to its length L. Thus a phase transition should occur when $\epsilon = Ts$, where s is the entropy per unit length and T is the temperature. Following these ideas, Nelson and Toner[6] have studied dislocation melting of a smectic-A liquid crystal starting from the Landau-Peierls[9] elastic energy for a system with one-dimensional mass-density waves. They showed that a smectic-A liquid crystal with a large and uniform density of dislocation lines behaves like nematic liquid crystal and suggested that the nematic—to—smectic-A transition can be described by dislocation melting of the smectic phase. They argued that this mechanism would lead to a anisotropic critical point with correlation-length exponents satisfying $\nu_{\parallel} = 2\nu_{\perp}$. Dasgupta and Halperin[8] have used a Villain form[10] similar to those studied by others[11] for a matter-lattice gauge theory describing type-II superconductors in three dimensions. They conclude from the model that the superconducting transition in three dimensions can be second order with xy exponents but inverted amplitudes, even though calculations based on the ϵ expansion[12] indicated a probable first-order transition. Finally, Toner[7] has formulated a soft-spin version of the Landau-Peierls dislocation Hamiltonian which he studied in an ϵ expansion. He found that this generalized model has a fixed point in $4-\epsilon$ dimensions with "inverted" xy exponents.

The de Gennes model[13] provides a phenomenological Hamiltonian for studying the nematic—to—smectic-A (NA) transition. This model established a strong analogy between smectic-A liquid crystals and superconductors and reduces to the Landau-Ginzburg model[14] for superconductors when the splay elastic constant K_1 is zero. The dislocation[15] models of smectic-A liquid crystals and superconductors studied to date do not make evident that connection between the two systems. A primary purpose of this paper is to derive a dislocation Hamiltonian for the smectic-A phase from the de Gennes free energy and to establish in detail the connection between dislocations in liquid crystals and vortices in superconductors. The dislocation Hamiltonian we derive will reduce to the superconducting vortex Hamiltonian when $K_1 = 0$ and will agree with the Landau-Peierls dislocation Hamiltonian for length scales large compared to penetration depths associated with twist and bend.

The elastic constant K_1 plays an important role[16–18] in determining the universality class of the

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NA transition. To all orders in perturbation theory one can show that the renormalization-group fixed point for the NA transition must satisfy (i) $K_1=0$, (ii) $K_1=\infty$, or (iii) $\nu_{\parallel}=(5-d)\nu_{\perp}$, where d is the spatial dimensionality. Furthermore, the fixed point at $K_1=0$ is stable and has the symmetry of the superconducting fixed point. Possible renormalization-group flows along K_1 are shown in Fig. 1. Within the ϵ and $1/n$ expansions[12,17,18] for a generalized de Gennes model with an $(n/2)$—complex-component order parameter, the only accessible fixed points are those with $K_1=0$ and $K_1=\infty$, the former being isotropic ($\nu_{\parallel}=\nu_{\perp}$) and the latter anisotropic ($\nu_{\parallel}>\nu_{\perp}$). The flow along the positive K_1 line is that shown in Fig. 1(a). There are fixed points satisfying $\nu_{\parallel}=(5-d)\nu_{\perp}$ in unphysical regions of the potential space in the ϵ expansion. Thus the ϵ and $1/n$ expansions provide no evidence for the Nelson-Toner

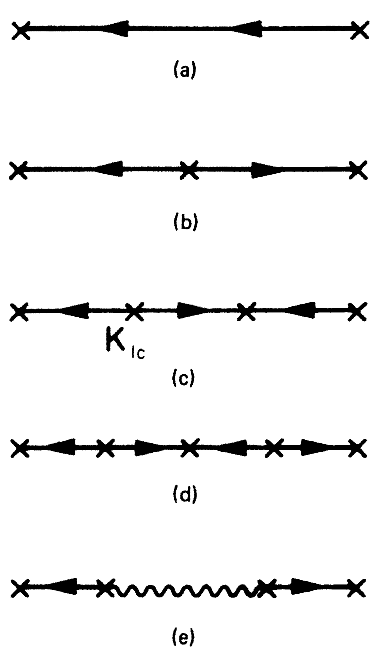

FIG. 1. Possible renormalization-group flows along the positive K_1 axis. Flow (a) with fixed points at $K_1=0$ and ∞ corresponds to the flow found to first order in $\epsilon=4-d$ and for all $2<d<4$ to first order in a $1/n$ expansion. Other possible flows are represented in (b)—(e). All fixed points with $0<K_1<\infty$ are anisotropic with $\nu_{\parallel}=(5-d)\nu_{\perp}$ ($\nu_{\parallel}=2\nu_{\perp}$ in three dimensions). Such fixed points exist in the ϵ expansion but with negative or complex values of $(K_1/K_3)^{1/2}$. Flow (c) corresponds to the most likely flow for the Nelson-Toner dislocation melting transition in three dimensions if it exists. Note that for this flow, there are two possible stable fixed points depending on the value of K_1. For $K_1<K_{1c}$, the transition is isotropic, whereas for $K_1>K_{1c}$ it is anisotropic. Flow (e) shows the possibility, which we cannot rule out, of a line of critical points depending on K_1, all with $\nu_{\parallel}=(5-d)\nu_{\perp}$.

$\nu_{\parallel}=2\nu_{\perp}$ transition in three dimensions. Since the Dasgupta-Halperin calculations provide strong evidence for a second-order transition in three dimensions when $K_1=0$, it is of some interest to study dislocation melting theories as a function of K_1 in the hope of providing evidence for the Nelson-Toner fixed point.

In this paper, we will study dislocation melting theories for the NA transition for $d=2$ and $d=4-\epsilon$ where there are well-controlled approximations. For $d=2$, we find that there is no NA transitions except at $T=0$ for $K_1<\infty$ in agreement with previous calculations.[4,19] For $K_1=\infty$, there is an NA transition in the *same* universality class as the Kosterlitz-Thouless[1,2] xy transition. For $K_1<\infty$, but for samples with linear dimensions smaller than some characteristic length there is a quasi—Kosterlitz-Thouless transition as in dirty superconductors.[4] In $4-\epsilon$ dimensions, we find an isotropic inverted xy fixed point with $K_1=0$ in agreement with Toner.[7] We also find an anisotropic fixed point with $K_1=\infty$ for the generalized n-component model that is identical to the $K_1=\infty$ fixed point[17] for the original de Gennes model. Thus in both two and $4-\epsilon$ dimensions there is no evidence for a $\nu_{\parallel}=(5-d)\nu_{\perp}$ fixed point. Clearly, more effort must be devoted to studying the physical case $n=2$, $d=3$.

II. ELASTICITY THEORIES

In order to present the various lengths that are relevant to the discussion of dislocations in smectics, we will begin our discussion with the complete de Gennes model.[13] The order parameter for the smectic-A phase is the complex amplitude ψ of a mass-density wave at wave number $q_0=2\pi/l$, where l is the layer spacing. The Frank director $\vec{n}(\vec{x})$ at position $\vec{x}$ is uniform in equilibrium with value $\vec{n}_0$. Deviations from equilibrium are expressed in terms of

$$\delta\vec{n}(\vec{x})=\vec{n}(\vec{x})-\vec{n}_0\ . \tag{2.1}$$

Since $\vec{n}(\vec{x})$ is a unit vector, $\delta\vec{n}(\vec{x})\cdot\vec{n}_0=0$ to lowest order in $\delta\vec{n}$. The de Gennes free energy in terms of ψ and $\delta\vec{n}$ is

$$\begin{aligned}F_0=\int d^dx[\,&a\,|\,\psi\,|^2+C_{\parallel}\,|\,\vec{\nabla}_{\parallel}\psi\,|^2\\&+C_{\perp}\,|\,(\vec{\nabla}_{\perp}-iq_0\delta\vec{n})\psi\,|^2\\&+\tfrac{1}{2}b\,|\,\psi\,|^4]\ ,\end{aligned} \tag{2.2}$$

where d is the dimensionality of space and $\parallel$ and $\perp$ refer to directions parallel and perpendicular to $\vec{n}_0$. The total free energy F for the smectic liquid crystal includes the Frank free energy F_F for the director

$$F=F_0+F_F\ ,$$
$$F_F=\tfrac{1}{2}\int d^3x\{K_1(\vec{\nabla}\cdot\vec{n})^2+K_2(\vec{n}\cdot\vec{\nabla}\times\vec{n})^2+K_3[\vec{n}\times(\vec{\nabla}\times\vec{n})]^2\}\ , \tag{2.3}$$

where for simplicity we have expressed F_F in three dimensions. Choosing $\vec{n}_0$ to be along the 1 axis and the wave number $\vec{q}$ to be in the 1-2 plane, we can express F_F to lowest order in $\delta\vec{n}$ in arbitrary dimension d as

$$F_F=\tfrac{1}{2}\int\frac{d^dq}{(2\pi)^d}[(K_1q_2^2+K_3q_1^2)\,|\,\delta\vec{n}_2\,|^2+(K_2q_2^2+K_3q_1^2)\,|\,\delta\vec{n}_t\,|^2]\ , \tag{2.4}$$

where $\delta\vec{n}_t$ is the projection of δn onto the $(d-2)$-dimensional subspace perpendicular to the 1-2 plane; $\delta\vec{n}_t=0$ for $d=2$, and $\delta n_t=\delta n_3$ for $d=3$.

The free energy defined in Eqs. (2.2) and (2.3) is highly anisotropic with many different lengths. There are two correlation lengths $\xi_{\parallel}=(C_{\parallel}/a)^{1/2}$ and $\xi_{\perp}=(C_{\perp}/a)^{1/2}$, and six penetration depths $\lambda_i=(K_i/q_0^2C_{\parallel}\,|\,\psi\,|^2)^{1/2}$ and $\bar{\lambda}_i=(K_i/q_0C_{\perp}\,|\,\psi\,|^2)^{1/2}$ for $i=1,2,3$. Our discussion can be greatly simplified by removing some anisotropy by a volume-preserving scale transformation

$$\begin{aligned} x_{\parallel}&\to(C_{\parallel}/C_{\perp})^{(1/2)(1-1/d)}x_{\parallel}\ ,\\ x_{\perp}&\to(C_{\parallel}/C_{\perp})^{-1/2d}x_{\perp}\ ,\\ \delta\vec{n}&\to(C_{\parallel}/C_{\perp})^{1/2d}\delta\vec{n}\ , \end{aligned} \tag{2.5}$$

which makes F_D isotropic,

$$F_D=\int d^dx[a\,|\,\psi\,|^2+C\,|\,(\vec{\nabla}+iq_0\delta\vec{n})\psi\,|^2+\tfrac{1}{2}b\,|\,\psi\,|^4]\ , \tag{2.6}$$

where $C=(C_{\parallel})^{1/d}(C_{\perp})^{1-1/d}$. The transformations of Eq. (2.5) leave the form of the Frank free energy invariant but renormalize the elastic constants via

$$\begin{aligned} K_{1,2}&\to(C_{\parallel}/C_{\perp})^{2/d}K_{1,2}\ ,\\ K_3&\to(C_{\parallel}/C_{\perp})^{-(1-2/d)}K_3\ ; \end{aligned} \tag{2.7}$$

with these changes, we only have one correlation length,

$$\xi=(C/a)^{1/2} \tag{2.8}$$

in mean-field theory and three penetration depths,

$$\lambda_i(K_i/B)^{1/2},\ \ i=1,2,3 \tag{2.9}$$

where

$$B=q_0^2C\,|\,\psi\,|^2\ . \tag{2.10}$$

λ_2 and λ_3 represent the real penetration depths of twist and bend into the smectic phase. λ_1, on the other hand, establishes the length scale associated with director splay in the smectic phase. It is clear that the de Gennes model reduces to the Landau-Ginzburg model for an isotropic superconductor in the limit $K_1\to 0$ and $K_2=K_3$, in which case there is only a single London penetration depth λ.

In the ordered phase, we can replace ψ by $|\,\psi\,|e^{iq_0u}$. The scale of spatial variations in ψ is set by ξ. Thus for spatial variations that are slow on this scale, we can write

$$\begin{aligned} F&=F_D'+F_F+\text{const}\ ,\\ F_D'&=\int d^dx[\tfrac{1}{2}B(\vec{\nabla}u-\delta\vec{n})^2]\ . \end{aligned} \tag{2.11}$$

The Landau-Peierls elastic energy for a one-dimensional solid follows from Eq. (2.11) by allowing $\delta\vec{n}$ to seek a value which minimizes F. We find

$$\begin{aligned} \delta n_2(\vec{q})&=-iq_2\frac{u(\vec{q})}{B+K_1q_2^2+K_3q_1^2}\ ,\\ \delta n_t(\vec{q})&=0\ , \end{aligned} \tag{2.12}$$

and

$$F_{\rm el}=\int\frac{d^dq}{(2\pi)^d}\left[\tfrac{1}{2}B\left[q_1^2+\frac{(\lambda_1^2q_2^2+\lambda_3^2q_1^2)q_2^2}{1+\lambda_1^2q_2^2+\lambda_3^2q_1^2}\right]\times|\,u(\vec{q})\,|^2\right]\ . \tag{2.13}$$

This equation describes the energy associated with variations in u when $q\xi<1$. If we now consider variations which are slow on a scale of λ_1 and λ_3 as well, we obtain the Landau-Peierls free energy

$$F_{\rm LP}=\tfrac{1}{2}\int d^dx[B(\vec{\nabla}_1u)^2+K_1(\nabla_2^2u)^2]\ . \tag{2.14}$$

The neglect of spatial variations of δn_t is also valid in this limit.

III. DISLOCATIONS

Dislocations are isolated points (in two dimensions) or lines (in three dimensions) where the order parameter ψ is zero and where the phase q_0u ceases to have meaning. Nevertheless, ψ must be single valued along any closed path Γ enclosing the dislocation. This limit changes in the phase q_0u in a complete circuit around Γ to integral multiples of 2π. Thus setting $\vec{v}=\vec{\nabla}u$, we have

$$\oint_{\Gamma}\vec{v}\cdot d\vec{s}=2\pi m/q_0=ml\ , \tag{3.1}$$

where m is an integer and $d\vec{s}$ is the differential of length along Γ. If there are many dislocations, we can introduce a source function $\vec{m}(\vec{x})$ in the usual way. In two dimensions, dislocations of strength

m_α are located at points $\vec{x}_\alpha$, and we have

$$\vec{m}(\vec{x}) = \sum_\alpha m_\alpha \hat{e}_3 \delta^{(2)}(\vec{x}-\vec{x}_\alpha) , \tag{3.2}$$

where $\hat{e}_3$ is the unit normal to the plane. In three dimensions, there are dislocation curves $\vec{R}_\alpha(s)$ forming closed loops or terminating at boundaries, and we have

$$\vec{m}(\vec{x}) = \sum_\alpha \int_\alpha \frac{d\vec{R}_\alpha}{ds} \delta^{(2)}(\vec{x}-\vec{R}_\alpha(s)) ds , \tag{3.3}$$

where $\delta^{(2)}$ is the two-dimensional δ function in the plane normal to $d\vec{R}_\alpha(s)/ds$. Expressing the right- and left-hand sides of Eq. (3.1) in terms of an integral over the covering surface Σ of Γ, we obtain

$$\vec{\nabla} \times \vec{v} = l\vec{m}(\vec{x}) . \tag{3.4}$$

In smectic-A liquid crystals, that nature of the dislocation depends on the orientation of $\vec{m}$ relative to $\vec{n}_0$. If $\vec{m}$ is parallel to $\vec{n}_0$, there is a screw dislocation, whereas if $\vec{m}$ is perpendicular to $\vec{n}_0$, there is an edge dislocation. For $\vec{m}$ pointing in an intermediate direction, the dislocation is neither pure screw nor pure edge. As we shall see shortly, the energies associated with edge and screw dislocations are different essentially because K_1 is nonzero. In two dimensions, it is obvious that there are only edge dislocations. In superconductors, the nature of vortices does not change with the direction of $\vec{m}$.

In the Appendix, we use Eq. (2.11) to calculate the energy of interacting dislocations. We find

$$E_{\text{disc}} = \tfrac{1}{2} \int d^dx\, d^dx' m_i(x) U_{ij}(x-x') m_j(x') + E_{\text{core}} , \tag{3.5}$$

where E_{core} is the core energy arising from changes in ψ within a distance ξ from the dislocation and where

$$U_{ij}(x) = \int \frac{d^dq}{(2\pi)^d} U_{ij}(q) e^{iq\cdot x} \tag{3.6}$$

with

$$U_{ij}(q) = l^2 B \left[\frac{K_1(q_2/q_1)^2+K_3}{[K_1(q_2/q_1)^2+K_3]q^2+B} e_{ti}e_{tj} + \frac{1}{q^2} \frac{K_2 q_2^2 + K_3 q_1^2}{(K_2 q_2^2+K_3 q_1^2)+B} e_{\perp i} e_{\perp j} \right]$$
$$\equiv U_e e_{ti} e_{tj} + U_s e_{\perp i} e_{\perp j} , \tag{3.7}$$

where

$$\vec{e}_t = \frac{\vec{n}_0 \times \vec{q}}{|\vec{n}_0 \times \vec{q}|} , \qquad \vec{e}_\perp = \frac{\vec{q} \times \vec{e}_t}{|\vec{q} \times \vec{e}_t|} . \tag{3.8}$$

Note that in two dimensions, both $\vec{n}_0$ and $\vec{q}$ are in the plane so that $\vec{e}_t = \vec{e}_3$ and only the first term of Eq. (3.7) is relevant. U_e describes energies of edge dislocations as can be seen by taking $\vec{m}$ to be perpendicular to $\vec{n}_0$. Similarly, U_s describes energies of screw dislocations as can be seen by taking $\vec{m}$ parallel to $\vec{n}_0$. It is clear that intermediate directions of $\vec{m}$ will yield contributions from both terms.

Equations (3.5)–(3.7) have several interesting features. First, when $K_1 \to 0$ and $K_2 = K_3 = K$, they reduce to the equations for the energy of interacting vortices in type-II superconductors. It is worth recalling that in this case, there is a self-energy per unit length ϵ_v, for a single vortex arising from the potential of Eq. (3.7),

$$\epsilon_v \sim \frac{Bl^2}{4\pi} \ln(\lambda^2/\xi^2), \quad \lambda \gg \xi . \tag{3.9}$$

Thus for $\lambda \gg \xi$, the energy per unit length of a vortex is dominated by the energy stored in the region within λ of the core. Second, the edge energy U_e is identical to that calculated using the Landau-Peierls free energy, provided that only the dominant momentum terms are kept. This implies setting $K_3 = 0$ and replacing q^2 by q_2^2 in Eq. (3.7). U_e is, however, more general than that obtained from the Landau-Peierls theory. We leave it as an exercise to the reader to derive U_e from the generalized Landau-Peierls energy of Eq. (2.13). In particular, it is valid for λ_1 and/or $\lambda_3 \to \infty$. In these cases the dislocations are no longer screened and $U_e \to Bl^2/q^2$. Finally, the energy per unit length ϵ_s for a screw dislocation with $K_1 = 0$ is identical to that for a vortex in a superconductor [Eq. (3.9)] with λ replaced by λ_2. Thus if one considers the core of a screw dislocation to extend out to a radius of λ_2 rather than ξ, one could say that all of the energy is in the core. This is consistent with the fact that there is no energy stored in the strain field $\vec{\nabla} u$ for a screw dislocation[15] if the Landau-Peierls elastic energy is used.

IV. PHASE TRANSITIONS

A. $d = 2$

In two dimensions, only edge dislocations survive. $U_e(\vec{x})$ can be calculated from Eqs. (3.6) and (3.7) when $\Lambda |\vec{x}| \gg 1$, where Λ is a cutoff of order ξ^{-1}. The form of $U_e(\vec{x})$ depends on the magnitude of $\vec{x}$ compared to the penetration depths λ_1 and λ_3. For small $\vec{x}$, we find

$$U_0(\vec{x}) = -\frac{Bl^2}{4\pi} \ln(|\vec{x}|^2/\bar{\lambda}^2) , \quad \text{for } \frac{x_1^2}{l_1^2} + \frac{x_2^2}{l_2^2} \ll 1 \tag{4.1}$$

where

$$\bar{\lambda}=2(\lambda_1+\lambda_3) \tag{4.2}$$

and

$$\begin{aligned} l_1^{-2}&=\frac{1}{4\lambda_3}\frac{(2\lambda_1+\lambda_3)}{(\lambda_1+\lambda_3)^2}\ , \\ l_2&=\bar{\lambda}\ . \end{aligned} \tag{4.3}$$

Thus U_e is logarithmic and isotropic in $\vec{x}$ for $\vec{x}$ within the ellipse defined in Eqs. (4.1) and (4.3). Samples whose spatial extent lies within this ellipse will exhibit a quasi–Kosterlitz-Thouless transition as in a dirty superconductor.[4] When λ_1 and/or $\lambda_3 \rightarrow \infty$, there will always be a Kosterlitz-Thouless transition. Note, however, that the ellipse becomes highly anisotropic if λ_1, rather than λ_3, becomes infinite.

For a large $\vec{x}$, the form of $U_e(\vec{x})$ depends on the angle $\vec{x}$ makes within the 1 axis. In region II of Fig. 2, $U_e(\vec{x})$ has the form

$$U_e^{(1)}=\tfrac{1}{4}Bl^2\left[\frac{\lambda_1}{\pi\,|x_1|}\right]^{1/2}\exp\left[-\frac{x_2^2}{4\lambda_1|x_1|}\right] \tag{4.4}$$

discussed by Toner and Nelson.[19] In region III, $U_e(\vec{x})$ dies off exponentially with distance with a characteristic length depending on λ_1 and λ_3. In both regions II and III, the bare vortex interaction is screened, and there is no phase transition except at $T=0$. Thus the nature of the NA transition in two dimensions depends on $\lambda_1=(K_1/B)^{1/2}$, but there is no evidence of a $\nu_{\parallel}=(5-d)\nu_{\perp}$ transition for any value of K_1.

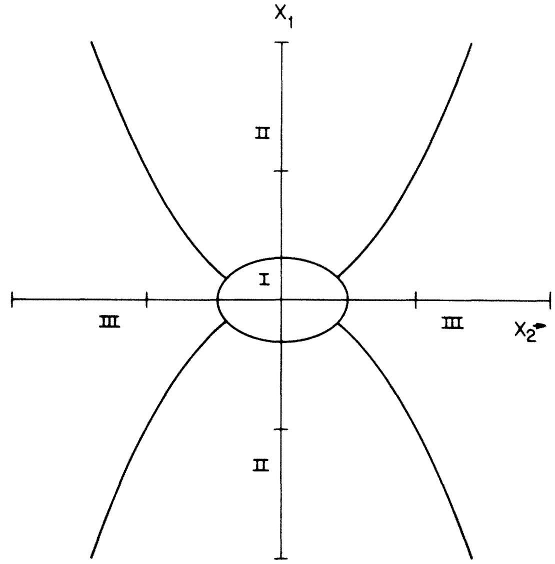

FIG. 2. Regions where asymptotic forms for $U_e(x_1,x_2)$ apply. In region I, U_e diverges logarithmically as $|\vec{x}|\rightarrow 0$; in region II it dies off as $|\vec{x}|^{-1/2}$ [cf. Eq. (4.4)], and in region III, it dies off exponentially.

B. Soft-spin models near $d=4$

In the Appendix, we review the steps that lead from a free energy for interacting dislocation loops to a Villain-type field theory. The reduced Hamiltonian for the soft-spin model generalized to an $(n/2)$–complex-component order parameter Ψ is then

$$\begin{aligned} H=\tfrac{1}{2}\int\frac{d^dq}{(2\pi)^d}A_i(\vec{q})\left[\left[\frac{q^2}{B}+\frac{1}{K_t(q)}\right]e_{ti}e_{tj}+\left[\frac{q^2}{B}+\frac{1}{K_{\perp}(q)}\right]e_{ti}e_{tj}\right]A_j(-\vec{q}) \\ +\int d^dx\left[r_0|\Psi|^2+\frac{U_0}{2}|\Psi|^4+|(\vec{\nabla}+i\lambda_0\vec{A})\Psi|^2\right], \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} K_{\perp}(q)&=K_1q_2^2/q_1^2+K_3\ , \\ K_t(q)&=K_2q_2^2/q_2+K_3q_1^2/q^2\ , \end{aligned} \tag{4.6}$$

and

$$\lambda_0=\frac{2\pi(k_BT)^{-1/2}}{q_0}\ . \tag{4.7}$$

Note the appearance to an effective coupling constant λ_0^2B which is proportional to q_0^{-2}, reflecting the nature of the duality transformation.

Since Ψ is related to the dual variables of the dislocation loops, disorder (order) in Ψ corresponds to order (disorder) in the original order parameter ψ. Thus any critical points in this model will have inverted amplitudes. We now consider the various fixed points of the Hamiltonian (4.5) in $4-\epsilon$ dimensions. The fixed points depend on the values of the elastic constants.

(1) *All K_i finite.* In this case, $\vec{A}$ is an irrelevant variable and can be integrated out leaving an effective Hamiltonian with n-vector symmetry. Thus there is a second-order transition with inverted xy

symmetry for the case $n=2$ in agreement with Toner.[7]

(2) $K_1=\infty$. In this case, the $d-2$ components of $\vec{A}_t$ are critical. The model then becomes exactly equivalent to the direct de Gennes model studied in Ref. 17. There is an anisotropic fixed point for $n>238.2$.

(3) $K_3=\infty$. All components of $\vec{A}$ are critical and the model is equivalent to the n-component superconductor studied in Ref. 12 and 17. Thus there are inverted superconducting fixed points for $n>365.9$. There is also a fixed point for $K_2=\infty$ which we will not consider in detail.

The recursion relation for K_1 for the Hamiltonian (4.5) is

$$\frac{dK_1}{dl}=\left[\epsilon+\frac{\nu_{\parallel}}{\nu_{\perp}}-3\right]K_1 \tag{4.8}$$

to all orders in perturbation theory. This differs from the equation

$$\frac{dK_1}{dl}=-\left[\epsilon+1-\frac{\nu_{\parallel}}{\nu_{\perp}}\right]K_1 \tag{4.9}$$

for K_1 in the original de Gennes model. This is because K_1 in the dual model is an elastic constant only in three dimensions when the two equations agree. In both models, recursion relations imply that $\nu_{\parallel}/\nu_{\perp}-1$ can be at most of order ϵ at any fixed point. Thus for the dual soft-spin models considered here, only the $K_1=\infty$ and $K_1=0$ fixed points can be studied. We conclude, therefore, that within the ϵ expansion for the soft-spin models, the flow along the K_1 line is again that shown in Fig. 1(a) and that there is no evidence for the Nelson-Toner fixed point.

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APPENDIX

In this appendix, we will derive the dislocation Hamiltonian discussed in Sec. III. We will then outline the derivation of the soft-spin Hamiltonian used in Sec. IV.

We begin by seeking the minimum energy configurations of F, Eq. (2.11). We find

$$\frac{\delta F}{\delta u}=B\vec{\nabla}\cdot(\nabla u-\delta\vec{n})=0\;, \tag{A1}$$

$$\frac{\delta F}{\delta n_i}=K_{ij}\delta n_j-B(\nabla_i u-\delta n_i)=0\;, \tag{A2}$$

where

$$\begin{aligned}K_{ij}=&-(K_1\nabla_2^2+K_3\nabla_1^2)e_{2i}e_{2j}\\&-(K_2\nabla_2^2+K_3\nabla_1^2)e_{1i}e_{1j}\;.\end{aligned} \tag{A3}$$

If there are dislocations present, then $\vec{\nabla}u=\vec{v}$ will have a singular part satisfying Eq. (3.4), and we can write

$$\vec{v}(\vec{q})=l\frac{i\vec{q}\times\vec{m}}{q^2}+i\vec{q}h(\vec{q})\;, \tag{A4}$$

where $h(\vec{q})$ is a function to be determined. Using Eq. (A1), we obtain

$$q^2h(\vec{q})=-i\vec{q}\cdot\delta\vec{n}\;, \tag{A5}$$

so that

$$\vec{v}=l\frac{i\vec{q}\times\vec{m}}{q^2}+\vec{q}\frac{\vec{q}\cdot\delta\vec{n}}{q^2}\;. \tag{A6}$$

Using this equation in Eq. (A2), we find

$$\begin{aligned}[(K_1q_2^2+K_3q_1^2)+Bq_1^2/q^2]\delta n_2&=iB\left[\frac{\vec{q}\times\vec{m}}{q^2}\right]_2\;,\\ [(K_2q_2^2+K_3q_1^2)+B]\delta n_3&=-B\left[\frac{\vec{q}\times\vec{m}}{q^2}\right]_3\;.\end{aligned} \tag{A7}$$

Finally, expressing all quantities in terms of $\vec{q}\times\vec{m}$ in Eq. (2.11), and using $(\vec{q}\times\vec{m})_2=\vec{e}_t\cdot\vec{m}$ and $(\vec{q}\times\vec{m})_3=\vec{e}_{\perp}\cdot\vec{m}$, we obtain Eqs. (3.5)–(3.7).

An alternate way to do this calculation would have been to change gauges via $u\rightarrow u+L$ and $\vec{A}=\delta\vec{n}+\vec{\nabla}L$. Then choosing the London gauge $\vec{\nabla}\cdot\vec{A}=0$, the calculations would have proceeded exactly as the usual ones for the vortex energy of a superconductor.[20] The energy for our system of interacting dislocation loops is

$$\begin{aligned}E=\tfrac{1}{2}\int\frac{d^dq}{(2\pi)^2}[\,(U_{ij}(\vec{q})+2E_c\delta_{ij})\\ \times m_i(\vec{q})m_j(\vec{q})]\;,\end{aligned} \tag{A8}$$

where $U_{ij}(\vec{q})$ is defined in Eq. (3.7), and E_c represents the core energy arising from variations within ξ of the core. Placing the dislocation cores on lattice sites $\vec{x}$, the partition function we wish to evaluate is

$$Z=\sum_{\vec{m}(\vec{x})}\delta(\vec{\Delta}\cdot\vec{m}(\vec{x}))e^{-\beta E}\;, \tag{A9}$$

where $\vec{\Delta}\cdot\vec{m}$ is the divergence of $\vec{m}$ on a lattice. Following Peskin[11] and Toner,[7] it is straightforward to convert Z into a Villain form. We obtain

$$Z=\int \prod d\vec{A}(\vec{x}) \int_{-\pi}^{\pi} \frac{d\phi(\vec{x})}{2\pi} \sum_{\vec{s}(\vec{x})} \left[\delta(\vec{\Delta}\cdot\vec{A}(\vec{x}))\exp[-\mathscr{H}_A(\vec{A}(\vec{x}))]\exp\left[-\frac{1}{4\beta E_c}\sum_{\vec{x}} |\lambda_0\vec{A}-\vec{\Delta}\phi+2\pi\vec{S}|\right]\right], \tag{A10}$$

where

$$\mathscr{H}_A=\tfrac{1}{2}\lambda_0^{-2}\int \frac{d^dq}{(2\pi)^d}\left[\left[\frac{q^2}{B}+\frac{1}{K_1q_2^2/q_1^2+K_3}\right]e_{ti}e_{tj}+\left[\frac{q^2}{B}+\frac{1}{K_2q_2^2/q^2+K_3q_1^2/q^2}\right]e_{\perp i}e_{\perp j}\right] \times A_i(\vec{q})A_j(-\vec{q}) . \tag{A11}$$

In Eq. (A10), $\vec{A}(\vec{x})$ is a vector field at site $\vec{x}$ and $\vec{S}(\vec{x})$ is a vector field with integer-valued components $S_i(\vec{x})$, $i=1,2,3$. Equation (A10) is very similar to the partition function derived by Toner. We note two differences between Eq. (10) and Toner's expression. First, in our formulation, the only constraint on $\vec{A}$ is that it be divergenceless, whereas Toner requires $A_3=0$ as well. This is because all of the energy of screw dislocations is contained in E_c in Toner's model and a Hubbard-Stratonovitch transformation for those components of $\vec{m}$ corresponds to screw dislocations is not necessary. Second, our expression for $\mathscr{H}_A$ has well-defined limits for K_1, K_2, or $K_3\to\infty$, whereas Toner's does not.

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